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The finite section method and problems in frame theory

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Abstract

The finite section method is a convenient tool for approximation of the inverse of certain operators using finite-dimensional matrix techniques. In this paper we demonstrate that the method is very useful in frame theory: it leads to an efficient approximation of the inverse frame operator and also solves related computational problems in frame theory. In the case of a frame which is localized w.r.t. an orthonormal basis we are able to estimate the rate of approximation. The results are applied to the reproducing kernel frame appearing in the theory for shift-invariant spaces generated by a Riesz basis.

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1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space. A family $\{f_k\}_{k=1}^{\infty}$ of elements in $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leqslant \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leqslant B\|f\|^2 \ \forall f \in \mathcal{H}.$$

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Given a frame $\{f_k\}_{k=1}^{\infty}$, the frame operator

$$S : \mathcal{H} \to \mathcal{H}, \ Sf = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k \tag{1}$$

is bounded and invertible, and each $f \in \mathcal{H}$ has the representation

$$f = \sum_{k=1}^{\infty} \langle S^{-1} f, f_k \rangle f_k, \tag{2}$$

see [4,6,15]. In order to use (2) in practice, we need efficient methods to invert the frame operator. The problem of designing finite-dimensional models for approximating the inverse frame operator leads to delicate questions of stability and convergence, cf. [5] and the references cited therein. In this paper we demonstrate that the finite section method, when applied properly, is very useful for this purpose.

We present the general results in Section 2. In Section 3 we apply our findings to two important issues in the theory of shift-invariant spaces generated by a Riesz basis: namely, inversion of the frame operator associated to the reproducing kernel frame, and reconstruction of a function from a set of sampling. Finally, in Section 4 we show that the finite section method leads to better results in general frame theory than the Casazza–Christensen method.

In the rest of this introduction we collect some basic facts concerning the finite section method. Let $\{f_k\}_{k=1}^{\infty}$ be a frame for a separable Hilbert space $\mathcal{H}$ and $\{\mathcal{H}_n\}_{n=1}^{\infty}$ a family of finite-dimensional subspaces of $\mathcal{H}$ for which

$$\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \cdots \subseteq \mathcal{H}_n \uparrow \mathcal{H}. \tag{3}$$

Let P_n denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_n$. Our purpose is to approximate a bounded operator $V : \mathcal{H} \to \mathcal{H}$ and its inverse. The basic definition, appearing in e.g., [10], is as follows.

Definition 1.1. Let $V : \mathcal{H} \to \mathcal{H}$ be a bounded operator, and assume that for each $n \in \mathbb{N}$ we have given a bounded operator $V_n : \mathcal{H}_n \to \mathcal{H}_n$.

(i) The sequence $\{V_n\}_{n=1}^{\infty}$ is an approximation method for the operator V if

$$V_n P_n f \to V f \qquad \text{for } n \to \infty \ \forall f \in \mathcal{H}.$$

(ii) An approximation method is applicable if there exists $n_0 \in \mathbb{N}$ such that for all $f \in \mathcal{H}$ the equation

$$V_n x = P_n f \tag{4}$$

has a unique solution x_n for all $n \geqslant n_0$, and x_n converges to a solution of the equation $Vx = f$.

(iii) The sequence $\{V_n\}_{n=1}^{\infty}$ is stable if there exists $n_o \in \mathbb{N}$ such that the operators V_n are invertible on $\mathcal{H}_n$ for $n \geqslant n_0$ and

$$\sup_{n \geqslant n_0} \|V_n^{-1} P_n\| < \infty.$$

We need two results from [10, Theorems 1.4, 1.17].

Lemma 1.2. *An approximation method* $\{V_n\}_{n=1}^{\infty}$ *associated to an operator V is applicable if and only if V is invertible and* $\{V_n\}_{n=1}^{\infty}$ *is stable.*

Lemma 1.2 implies that if $\{V_n\}_{n=1}^{\infty}$ is applicable, then V is invertible and

$$V_n^{-1} P_n f \to V^{-1} f \ \forall f \in \mathcal{H}.$$

Lemma 1.3. *Assume that the approximation method* $\{V_n\}_{n=1}^{\infty}$ *is stable. If* $\{W_n\}_{n=1}^{\infty}$ *is a sequence of operators for which*

$$\limsup_{n\to\infty} \|W_n P_n\| < \liminf_{n\to\infty} \|V_n^{-1} P_n\|^{-1},$$

then $\{V_n + W_n\}_{n=1}^{\infty}$ *is stable.*

An example of an approximation method associated to V is the family of operators $\{P_n V P_n\}_{n=1}^{\infty}$, where P_n are orthogonal projections onto subspaces $\mathcal{H}_n$ satisfying (3). This special type of approximation method is called a *finite section method*.

Proposition 1.4. *Assume that* $\{\mathcal{H}_n\}_{n=1}^{\infty}$ *is a sequence of closed subspaces of* $\mathcal{H}$ *satisfying* (3)*, and let* P_n *be the orthogonal projection onto* $\mathcal{H}_n$*. Then the finite section method applies for an arbitrary positive definite operator V.*

Proposition 1.4 is proved in [10, p. 32], for the case where there is an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ such that $\mathcal{H}_n = \text{span}\{e_k\}_{k=1}^{n}$. The general case follows from here. We note that for an arbitrary invertible operator V there always exists an orthonormal basis such that the finite section method applies. However, for practical purposes the pure existence is not enough: we need to know which basis to use. Since all operators appearing in the sequel are positive definite, we avoid this complication.

It is usually most convenient, in particular from a numerical viewpoint, to use the finite section method in its matrix version:

Remark 1.5. The matrix formulation of the equation $Vx = f$ with respect to an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ is

$$\begin{bmatrix} \langle Ve_1, e_1\rangle & \langle Ve_2, e_1\rangle & \cdot & \cdot & \cdot \\ \langle Ve_1, e_2\rangle & \langle Ve_2, e_2\rangle & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \langle x, e_1\rangle \\ \langle x, e_2\rangle \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \langle f, e_1\rangle \\ \langle f, e_2\rangle \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}.$$

In case $\mathcal{H}_n$ has the orthonormal basis $\{e_k\}_{k=1}^{n}$, the matrix version of the Eq. (4) w.r.t. the finite section method,

$$P_n V P_n x_n = P_n f \tag{5}$$

is

$$\begin{bmatrix} \langle Ve_1, e_1\rangle & \langle Ve_2, e_1\rangle & \cdot & \cdot & \langle Ve_n, e_1\rangle \\ \langle Ve_1, e_2\rangle & \langle Ve_2, e_2\rangle & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \langle Ve_1, e_n\rangle & \langle Ve_2, e_n\rangle & \cdot & \cdot & \langle Ve_n, e_n\rangle \end{bmatrix} \begin{bmatrix} \langle x_n, e_1\rangle \\ \langle x_n, e_2\rangle \\ \cdot \\ \cdot \\ \langle x_n, e_n\rangle \end{bmatrix} = \begin{bmatrix} \langle f, e_1\rangle \\ \langle f, e_2\rangle \\ \cdot \\ \cdot \\ \langle f, e_n\rangle \end{bmatrix}.$$

If the finite section method applies, this finite matrix equation has a unique solution for n sufficiently large,

$$\begin{bmatrix} \langle x_n, e_1\rangle \\ \langle x_n, e_2\rangle \\ \cdot \\ \cdot \\ \langle x_n, e_n\rangle \end{bmatrix} = \begin{bmatrix} \langle Ve_1, e_1\rangle & \langle Ve_2, e_1\rangle & \cdot & \cdot & \langle Ve_n, e_1\rangle \\ \langle Ve_1, e_2\rangle & \langle Ve_2, e_2\rangle & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \langle Ve_1, e_n\rangle & \langle Ve_2, e_n\rangle & \cdot & \cdot & \langle Ve_n, e_n\rangle \end{bmatrix}^{-1} \begin{bmatrix} \langle f, e_1\rangle \\ \langle f, e_2\rangle \\ \cdot \\ \cdot \\ \langle f, e_n\rangle \end{bmatrix},$$

and (5) has the solution

$$x_n = \sum_{k=1}^{n} \langle x_n, e_k\rangle e_k.$$

Furthermore $x_n \to V^{-1} f$.

We end this section by Schur's Lemma, [4,15], which will be needed repeatedly.

Lemma 1.6. *Let* $M = \{M_{j,k}\}_{j,k=1}^{\infty}$ *be a matrix for which* $M_{j,k} = \overline{M_{k,j}}$ *for all* $j, k \in \mathbb{N}$ *and for which there exists a constant* $B > 0$ *such that*

$$\sum_{k=1}^{\infty} |M_{j,k}| \leqslant B \; \forall j \in \mathbb{N}.$$

Then M defines a bounded operator on $\ell^2(\mathbb{N})$ *of norm at most B.*

2. Approximation of the inverse frame operator

Given a frame $\{f_k\}_{k=1}^{\infty}$, we now consider approximation of the frame operator S defined in (1). The frame operator is positive definite, so the finite section method applies for all families of projection operators P_n on spaces $\mathcal{H}_n$ satisfying (3). However, in order to proceed, we need an easily computable form of the operators $P_n S P_n$, which in practice is not always available due to the fact that the frame operator is defined via an infinite series. In order to develop a practically useful method we have to replace the operators $P_n S P_n$ by some operators which can be found using only finite-dimensional linear algebra.

Our method is based on the following results:

Lemma 2.1. *Assume that* $\{\mathcal{H}_n\}_{n=1}^{\infty}$ *is a sequence of closed subspaces of* $\mathcal{H}$ *satisfying* (3), *and let* P_n *be the orthogonal projection onto* $\mathcal{H}_n$. *Consider* P_nSP_n *as an operator on* $\mathcal{H}_n$. *Then* P_nSP_n *is invertible and self-adjoint*; *letting* I_n *denote the identity on* $\mathcal{H}_n$, *we have*

$$AI_n \leqslant P_nSP_n \leqslant BI_n, \ \frac{1}{B} I_n \leqslant (P_nSP_n)^{-1} \leqslant \frac{1}{A} I_n.$$

In particular,

$$\left\| (P_nSP_n)^{-1} \right\| \leqslant \frac{1}{A}.$$

Proof. It is clear that P_nSP_n is self-adjoint. Given $f \in \mathcal{H}_n$,

$$\langle P_nSP_nf, f\rangle = \langle Sf, f\rangle = \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2.$$

If $P_nSP_nf = 0$ for some $f \in \mathcal{H}_n$ it follows from here that $f = 0$. The rest follows from the frame condition. □

Lemma 2.2. *Assume that for each* $n \in \mathbb{N}$ *we have an operator* $\Lambda_n : \mathcal{H}_n \to \mathcal{H}_n$ *for which*

$$\|\Lambda_n - P_nSP_n\| \to 0 \text{ as } n \to \infty. \tag{6}$$

Then

$$\Lambda_nP_nf \to Sf \text{ as } n \to \infty \ \forall f \in \mathcal{H}.$$

Furthermore, the sequence $\{\Lambda_n\}_{n=1}^{\infty}$ *is stable*, Λ_n *is invertible for sufficiently large values of* $n \in \mathbb{N}$, *and*

$$\Lambda_n^{-1}P_nf \to S^{-1}f \text{ as } n \to \infty \ \forall f \in \mathcal{H}.$$

Proof. By Lemma 2.1,

$$\liminf_{n\to\infty} \left\| (P_nSP_n)^{-1}P_n \right\|^{-1} \geqslant A.$$

Now Lemma 1.3 implies that the sequence $\{\Lambda_n\}_{n=1}^{\infty}$ is stable, and Lemma 1.2 gives the rest. □

We have already seen that if $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $\mathcal{H}_n$ =span$\{e_k\}_{k=1}^{n}$, then the jl-th entry in the matrix for P_nSP_n with respect to $\{e_k\}_{k=1}^{n}$ is $\langle Se_l, e_j\rangle$. For each $n \in \mathbb{N}$ we now give conditions on an $n \times n$ matrix $\{\lambda_{j,l}^n\}_{j,l=1}^{n}$ which imply that the corresponding operators Λ_n satisfy (6).

Lemma 2.3. *Fix* $\varepsilon > 0$. *Let* $n \in \mathbb{N}$, *and let* $\{\lambda_{j,l}^n\}_{j,l=1}^n$ *be a hermitian matrix such that*

$$\left|\langle Se_l, e_j\rangle - \lambda_{j,l}^n\right| \leqslant \frac{\varepsilon}{n}\, 2^{-|j-l|}, \;\; j,l = 1,\ldots,n. \tag{7}$$

Then

$$\|\Lambda_n - P_n S P_n\| \leqslant \frac{3\varepsilon}{n}.$$

Proof. Via Schur's Lemma the norm of the operator given by the matrix $\{\langle Se_l, e_j\rangle - \lambda_{j,l}^n\}_{j,l=1}^n$ can be estimated by

$$\left\|\{\langle Se_l, e_j\rangle - \lambda_{j,l}^n\}_{k,l=1}^n\right\| \leqslant \frac{1}{n}\left\|\begin{bmatrix} \varepsilon & \varepsilon/2 & \cdot & \cdot & \varepsilon/2^n \\ \varepsilon/2 & \varepsilon & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \varepsilon/2^n & \cdot & \cdot & \cdot & \varepsilon \end{bmatrix}\right\| \leqslant \frac{3\varepsilon}{n}. \qquad \square$$

It is very natural to let the matrix entries $\lambda_{j,l}^n$ appearing in (7) be related to the partial sums of the sum defining the frame operator. Note that for $n \in \mathbb{N}$, the frame operator associated with $\{f_k\}_{k=1}^n$ is

$$S_n : \text{ span }\{f_k\}_{k=1}^n \to \text{span}\{f_k\}_{k=1}^n, \;\; S_n f = \sum_{k=1}^n \langle f, f_k\rangle f_k. \tag{8}$$

We will choose $\lambda_{j,l}^n$ of the form

$$\lambda_{j,l}^n = \langle S_{m(n)} e_l, e_j\rangle, \;\; j,l = 1,\ldots,n; \tag{9}$$

here we have to find the number $m(n) \geqslant n$ such that (7) is satisfied. This can *always* be done. We show in Lemmas 2.5 and 2.6 how explicit values for $m(n)$ can be obtained for *localized frames*, a concept introduced by Gröchenig in [8].

Definition 2.4. The frame $\{f_k\}_{k=1}^\infty$ is *polynomially localized* with respect to the orthonormal basis $\{e_k\}_{k=1}^\infty$ with decay $s > 0$ (or simply s-localized), if for some constant $C > 0$

$$|\langle f_k, e_l\rangle| \leqslant C(1 + |k-l|)^{-s} \qquad \forall k, l \in \mathbb{N}. \tag{10}$$

The frame $\{f_k\}_{k=1}^\infty$ is *exponentially localized* with respect to the orthonormal basis $\{e_k\}_{k=1}^\infty$ if for some $\alpha > 0$ and some constant $C > 0$

$$|\langle f_k, e_l\rangle| \leqslant C e^{-\alpha|k-l|} \qquad \forall k, l \in \mathbb{N}. \tag{11}$$

Localization with respect to a Riesz basis $\{e_k\}_{k=1}^\infty$ is defined similarly, except that condition (10) (resp. (11)) also is assumed to hold with $\{e_k\}_{k=1}^\infty$ replaced by the dual Riesz basis $\{\tilde{e}_k\}_{k\in\mathbb{Z}}$.

Lemma 2.5. *Assume that the frame* $\{f_k\}_{k=1}^{\infty}$ *is exponentially localized with respect to the basis* $\{e_k\}_{k=1}^{\infty}$. *Then, for any* $m(n)\geqslant n$,

$$\left\| \{|\langle Se_l, e_j\rangle - \langle S_{m(n)}e_l, e_j\rangle|\}_{j,l=1}^{n} \right\| \leqslant C^2 \frac{e^{-2\alpha}}{(1-e^{-2\alpha})(1-e^{-\alpha})}\, e^{-2\alpha(m(n)-n)}.$$

Proof. If we choose $m(n)\geqslant n$, then for all $j, l = 1, \dots, n$,

$$\begin{aligned} |\langle Se_l, e_j\rangle - \langle S_{m(n)}e_l, e_j\rangle| &= \left| \sum_{k=m(n)+1}^{\infty} \langle e_l, f_k\rangle\langle f_k, e_j\rangle \right| \\ &\leqslant C^2 \sum_{k=m(n)+1}^{\infty} e^{-\alpha|k-l|}e^{-\alpha|k-j|} \\ &\leqslant C^2 \sum_{k=m(n)+1}^{\infty} e^{-\alpha(2k-l-j)} \\ &= C^2 e^{-2\alpha(m(n)+1)+\alpha(l+j)}(1-e^{-2\alpha})^{-1}. \end{aligned} \tag{12}$$

Thus, the entries in the matrix $\{|\langle Se_l, e_j\rangle - \langle S_{m(n)}e_l, e_j\rangle|\}_{j,l=1}^{n}$ are element-wise smaller than or equal to the entries in the matrix

$$C^2 \frac{e^{-2\alpha(m(n)+1)}}{1-e^{-2\alpha}} \begin{bmatrix} e^{2\alpha} & e^{3\alpha} & \cdot & \cdot & e^{\alpha(n+1)} \\ e^{3\alpha} & e^{4\alpha} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ e^{\alpha(n+1)} & e^{\alpha(n+2)} & \cdot & \cdot & e^{2n\alpha} \end{bmatrix}$$

$$= C^2 \frac{e^{-2\alpha(m(n)-n)}}{1-e^{-2\alpha}} \begin{bmatrix} e^{-2\alpha n} & e^{-\alpha(2n-1)} & \cdot & \cdot & e^{-\alpha(n+1)} \\ e^{-\alpha(2n-1)} & e^{-\alpha(2n-2)} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ e^{-\alpha(n+1)} & e^{-\alpha n} & \cdot & \cdot & e^{-2\alpha} \end{bmatrix}.$$

The norm of this matrix can be estimated by Schurs Lemma, so we arrive at

$$\begin{aligned} \left\| \{|\langle Se_l, e_j\rangle - \langle S_{m(n)}e_l, e_j\rangle|\}_{j,l=1}^{n} \right\| &\leqslant C^2 \frac{e^{-2\alpha(m(n)-n)}}{1-e^{-2\alpha}} \sum_{k=2}^{n+1} e^{-\alpha k} \\ &\leqslant C^2 \frac{e^{-2\alpha}}{(1-e^{-2\alpha})(1-e^{-\alpha})}\, e^{-2\alpha(m(n)-n)}. \qquad \square \end{aligned}$$

The result in Lemma 2.5 can be formulated slightly differently: in fact, by choosing $m(n) = rn$ for some $r > 1$, there exists a constant $C' > 0$ such that

$$\left\| \{|\langle Se_l, e_j\rangle - \langle S_{m(n)}e_l, e_j\rangle|\}_{j,l=1}^{n} \right\| \leqslant C' e^{-2\alpha(r-1)n}.$$

In words, this says that exponential localization of the frame leads to an exponential rate of approximation of $\{\langle Se_l, e_j\rangle\}_{j,l=1}^{n}$.

Lemma 2.6. *Assume that the frame* $\{f_k\}_{k=1}^{\infty}$ *is s-localized with respect to the basis* $\{e_k\}_{k=1}^{\infty}$, *with* $s > 1$. *If* $m(n) \geqslant (2n)^{\frac{s}{s-1}}$ *then*

$$\left\| \{|\langle Se_l, e_j\rangle - \langle S_{m(n)}e_l, e_j\rangle|\}_{j,l=1}^{n} \right\| \leqslant C' n^{-s}.$$

Proof. We have

$$\left|\langle Se_l, e_j\rangle - \langle S_{m(n)}e_l, e_j\rangle\right| = \left| \sum_{k=m(n)+1}^{\infty} \langle e_l, f_k\rangle\langle f_k, e_j\rangle \right| \tag{13}$$

$$\leqslant C^2 \sum_{k=m(n)+1}^{\infty} (1+|k-l|)^{-s}(1+|k-j|)^{-s}. \tag{14}$$

We define the index sets $I_1 = \{k \geqslant m(n)+1 : |k-l| \leqslant |l-j|/2\}$ and $I_2 = \{k \geqslant m(n)+1 : |k-l| > |l-j|/2\}$. Proceeding as in [11] we split the sum in (14) into two sums, such that k runs through the index set I_1 and I_2, respectively. If $k \in I_1$ then $|k-j| \geqslant |j-l|/2$, hence

$$C^2 \sum_{k \in I_1} (1+|k-l|)^{-s}(1+|k-j|)^{-s} \leqslant C^2(1+|j-l|/2)^{-s} \sum_{k \in I_1} (1+|k-l|)^{-s}.$$

Furthermore

$$C^2 \sum_{k \in I_2} (1+|k-l|)^{-s}(1+|k-j|)^{-s} \leqslant C^2(1+|j-l|/2)^{-s} \sum_{k \in I_2} (1+|k-j|)^{-s}.$$

Thus

$$\begin{aligned} &\left| \sum_{k=m(n)+1}^{\infty} \langle e_l, f_k\rangle\langle f_k, e_j\rangle \right| \\ &\quad \leqslant 2^s C^2 (1+|l-j|)^{-s} \left(\sum_{k=m(n)+1}^{\infty} (1+|k-l|)^{-s} + \sum_{k=m(n)+1}^{\infty} (1+|k-j|)^{-s} \right) \\ &\quad \leqslant \frac{2^s C^2}{s-1} (1+|l-j|)^{-s} \left[\left(m(n)+1-l)^{-s+1}\right) + \left(m(n)+1-j)^{-s+1}\right) \right], \end{aligned}$$

where we have used the estimate

$$\sum_{k=m(n)+1}^{\infty} (1+(k-l))^{-s} \leqslant \int_{m(n)}^{\infty} (1+x-l)^{-s} dx = \frac{(m(n)+1-l)^{-s+1}}{s-1}.$$

By assumption $m(n) \geqslant (2n)^{\frac{s}{s-1}}$, so for $r = 1, \ldots n$,

$$(m(n)+1-r)^{-s+1} \leqslant \left((2n)^{\frac{s}{s-1}} + 1 - r\right)^{-s+1} \leqslant \left(n^{\frac{s}{s-1}}\right)^{-s+1} = n^{-s}.$$

Using that $\sup_j \sum_{l=1}^{\infty}(1+|l-j|)^{-s} < \infty$ (by [8, Lemma 2.1]) and applying Schur's Lemma we obtain

$$\left\| \{|\langle Se_l, e_j\rangle - \langle S_{m(n)}e_l, e_j\rangle|\}_{j,l=1}^{n} \right\| \leqslant \frac{2^s C^2}{s-1}\, n^{-s} \sum_{l=1}^{n} (1+|l-j|)^{-s} \leqslant C' n^{-s}.$$

as claimed. □

In the two previous lemmas we have shown that $\langle Se_l, e_j\rangle$ can be approximated by $\langle S_{m(n)}e_l, e_j\rangle$ with an error rate that depends on the localization of the frame $\{f_k\}_{k=1}^{\infty}$. However in practice we may not even know the matrix entries $\lambda_{j,l}^n = \langle S_{m(n)}e_l, e_j\rangle$ exactly. When we compute the inner products $\langle S_{m(n)}e_l, e_j\rangle$ by numerical integration we obtain an approximation $\tilde{\lambda}_{j,l}^n$ to $\lambda_{j,l}^n$, hence we introduce an additional error. But for "reasonable" functions $\{f_k\}_{k=1}^{\infty}, \{e_k\}_{k=1}^{\infty}$ it is not difficult to approximate $\lambda_{j,l}^n$ by standard numerical integration techniques such that the error $|\tilde{\lambda}_{j,l}^n - \lambda_{j,l}^n|$ is always smaller than any prescribed tolerance. Thus we will henceforth tacitly assume that the matrix entries of $\{\lambda_{j,l}^n\}$ have been computed with sufficient accuracy and absorb any error resulting from numerical integration in a constant in our error estimates.

We now show how the localization properties of a given frame determine the convergence order of the proposed approximation method. We need the following result; (a) and (b) follows from Jaffard's "lemmes de la fenêtre" in Section III of [11], and (c) is a classical result which can be found, e.g., in [12].

Lemma 2.7. *Let $A = [A_{k,l}]$ and $B = [B_{k,l}]$ be two invertible matrices with $k, l \in \mathbb{N}$ and let A_n and B_n be $n \times n$ principal leading submatrices of A and B, respectively. Assume that there exists an $n_0 \in \mathbb{N}$ such that A_n and B_n are invertible for all $n \geqslant n_0$.*

(a) *If there exist $C, \alpha > 0$ such that for all $n \geqslant n_0$*

$$|[A_n]_{k,l} - [B_n]_{k,l}| \leqslant C e^{-\alpha|k-l|},$$

then there exist $C_1, \alpha_1 > 0$ independent of n such that for all $n \geqslant n_0$

$$|[A_n^{-1}]_{k,l} - [B_n^{-1}]_{k,l}| \leqslant C_1 e^{-\alpha_1|k-l|}.$$

(b) *If there exist $C > 0, s > 1$ such that for all $n \geqslant n_0$*

$$|[A_n]_{k,l} - [B_n]_{k,l}| \leqslant C(1+|k-l|)^{-s},$$

then there exists a $C_1 > 0$ independent of n such that for all $n \geqslant n_0$

$$|[A_n^{-1}]_{k,l} - [B_n^{-1}]_{k,l}| \leqslant C_1(1+|k-l|)^{-s}.$$

(c) *If there exist $C, \alpha > 0$ such that*

$$|A_{k,l}| \leqslant C e^{-\alpha|k-l|} \ \forall k, l,$$

then there exist $C_1, \alpha_1 > 0$ such that

$$|[A^{-1}]_{k,l}| \leqslant C_1 e^{-\alpha_1|k-l|} \ \forall k, l.$$

Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. We define the operators T, T_n via

$$T : \ell^2 \to \mathcal{H},\ T\{c_k\} = \sum_{k=1}^{\infty} c_k e_k, \quad T_n : \mathbb{C}^n \to \mathcal{H},\ T_n\{c_k\} = \sum_{k=1}^{n} c_k e_k.$$

Their adjoints are given by

$$T^* : \mathcal{H} \to \ell^2,\ T^* f = \{\langle f, e_k\rangle\}_{k=1}^{\infty}, \quad T_n^* : \mathcal{H} \to \mathbb{C}^n, T_n^* f = \{\langle f, e_k\rangle\}_{k=1}^{n}.$$

For $n \in N$ and $x \in \ell^2(\mathbb{Z})$ we define the orthogonal projections P_n by

$$P_n x = (x_1, x_2, \ldots, x_n, 0, 0, \ldots), \tag{15}$$

and identify the image of P_n with the n-dimensional space $\mathbb{C}^n$ (in this sense $T_n P_n\{c_k\}_{k=1}^{n} = T_n\{c_k\}_{k=1}^{n}$).

Lemmas 2.5 and 2.6 tell explicitly (in terms of the involved constants) how to choose $m(n) > n$ such that (7) is satisfied; by Lemmas 2.3 and 2.2 this implies that the $n \times n$ matrix Λ_n with entries

$$\lambda_{j,l}^{n} = \langle S_{m(n)} e_l, e_j\rangle,\ \ j, l = 1, \ldots, n \tag{16}$$

is invertible for n sufficiently large. In the formulation of Theorem 2.8 below we tacitly assume that n is chosen sufficiently large.

Theorem 2.8. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame with frame operator S and let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Furthermore, let Λ_n be the $n \times n$ matrix with the entries defined in* (16). *Let $f \in \mathcal{H}$ and set*

$$h_n = \sum_{k=1}^{n} (\Lambda_n^{-1} T_n^* f)_k e_k.$$

(a) *Assume that $\{f_k\}_{k=1}^{\infty}$ is exponentially localized w.r.t. $\{e_k\}_{k=1}^{\infty}$ and that there exists a constant $C > 0$ and an $\alpha > 0$ such that*

$$|\langle f, e_k\rangle| \leqslant C e^{-\alpha k} \quad \text{for } k \in \mathbb{N}.$$

If we choose $m(n) = rn$ for some $r > 1$, then for $n_0 \in \mathbb{N}$ large enough

$$\|S^{-1} f - h_n\| \leqslant C' e^{-\alpha' n} \qquad \text{for all } n > n_0,$$

for some $\alpha' > 0$ (but possibly $\alpha' < \alpha$) and some constant $C' > 0$ independent of n.

(b) *Assume that $\{f_k\}_{k=1}^{\infty}$ is s-localized w.r.t. $\{e_k\}_{k=1}^{\infty}$ and that there exists a constant $C > 0$ such that*

$$|\langle f, e_k\rangle| \leqslant C(1 + k)^{-s} \qquad \text{for } k \in \mathbb{N}.$$

If $m(n)$ is chosen as in Lemma 2.6 *then for $n_0 \in \mathbb{N}$ large enough*

$$\|S^{-1}f - h_n\| \leqslant C'(1+n)^{-s+1} \qquad \text{for all } n > n_0,$$

for some constant $C' > 0$ independent of n.

Proof. We only show part (a), the proof of part (b) is similar. In what follows α_k and C_k denote positive constants, with C_k depending on α_k, but both constants independent of n. It is a consequence of Lemma 2.2 and the choice of $m(n)$ that there exists an n_0 such that for all $n > n_0$ the matrix Λ_n is invertible. Let Λ be the matrix given by $\Lambda_{j,l} = \langle Se_l, e_j\rangle$. The reader will easily convince herself that $P_n\Lambda P_n$ is invertible, since Λ is hermitian positive definite. For $n > n_0$ we estimate

$$\begin{aligned}\|S^{-1}f - h_n\| &= \|T\Lambda^{-1}T^*f - T_n\Lambda_n^{-1}T_n^*f\| \\ &\leqslant \|T\Lambda^{-1}T^*f - T_nP_n\Lambda^{-1}T^*f\| \\ &\quad + \|T_nP_n\Lambda^{-1}T^*f - T_n(P_n\Lambda P_n)^{-1}P_nT^*f\| \\ &\quad + \|T_n(P_n\Lambda P_n)^{-1}P_nT^*f - T_n\Lambda_n^{-1}T_n^*f\|. \end{aligned} \tag{17}$$

We estimate the three terms on the right-hand-side of (17) separately.

Since $\langle Se_l, e_j\rangle = \sum_{k=1}^{\infty}\langle e_l, f_k\rangle\langle f_k, e_j\rangle$ and since $\{f_k\}_{k=1}^{\infty}$ is exponentially localized by assumption we can apply Proposition 3.4(b) in [8] and conclude that $|\Lambda_{k,l}| \leqslant C_0 e^{-\alpha_0|k-l|}$. By Proposition 2 in [11] it follows that the entries of Λ^{-1} satisfy $|\Lambda_{k,l}^{-1}| \leqslant C_1 e^{-\alpha_1|k-l|}$. Hence

$$\|T\Lambda^{-1}T^*f - T_nP_n\Lambda^{-1}T^*f\| = \left\| \sum_{k=n+1}^{\infty} (\Lambda^{-1}\{\langle f, e_l\rangle\}_{l=1}^{\infty})_k e_k \right\| \leqslant C_2 e^{-\alpha_2 n}. \tag{18}$$

Concerning the second term on the right-hand-side of (17) we recall that Λ is a hermitian positive-definite matrix. It is well-known that the finite section method applies in this case (see e.g. [14, Lemma 2.3]) and thus $(P_n\Lambda P_n)^{-1}P_nT^*f \to \Lambda^{-1}T^*f$ as $n \to \infty$. Since exponential off-diagonal decay of Λ implies exponential off-diagonal decay of Λ^{-1} (see Lemma 2.7(c)) and since T^*f decays exponentially by assumption, we can proceed along the same lines as in the proof of (3.18) of Theorem 3 in [13] and obtain

$$\|T_n\Lambda^{-1}T^*f - T_n(P_n\Lambda P_n)^{-1}P_nT^*f\| \leqslant \|T_n\|C_3e^{-\alpha_3 n} \leqslant C_3 e^{-\alpha_3 n}. \tag{19}$$

Let L_n denote the $n \times n$ matrix $[\Lambda_{k,l}]_{k,l=1}^{n}$. It is easy to see that

$$T_n(P_n\Lambda P_n)^{-1}P_nT^*f = T_nL_n^{-1}T_n^*f. \tag{20}$$

Hence, since $\|\Lambda_n^{-1}\| \to \|\Lambda^{-1}\|$ as $n \to \infty$ (by Lemmas 2.2 and 2.7), $\|L_n^{-1}\| \leqslant \|\Lambda^{-1}\|$, and $\|\Lambda_n - L_n\| \leqslant C_4 e^{-\alpha_4 n}$ (by (2)) there holds

$$\begin{aligned}\|T_n(P_n\Lambda P_n)^{-1}P_nT^*f - T_n\Lambda_n^{-1}T_n^*f\| &= \|T_nL_n^{-1}T_n^*f - T_n\Lambda_n^{-1}T_n^*f\| \\ \leqslant \|T_n\|\|L_n^{-1}\|\|\Lambda_n - L_n\|\|\Lambda_n^{-1}\|\|T_n^*f\| &\leqslant C_5e^{-\alpha_5 n}. \end{aligned} \tag{21}$$

Combining (17) with estimates (18), (19), and (21) yields Theorem (2.8) (a). □

We note that Gröchenig recently introduced the concept intrinsically localized frames in [9]; the advantage compared to the type of localization discussed in [8] that the definition is given directly in terms of the elements in the frame, i.e., no choice of an orthonormal basis needs to be made. We expect results similar to Theorem 2.8 to hold for intrinsically localized frames, but it is not immediately clear how to modify the proof.

3. Frames for shift-invariant spaces

In this section we demonstrate that the proposed method can be used to compute numerically dual frames for frames related to shift-invariant spaces. Throughout this section it is more convenient to use $\mathbb{Z}$ rather than $\mathbb{N}$ as index set for the frame elements. It is easy to see that all results in Sections 1 and 2 can be reformulated for frames of the form $\{f_k\}_{k\in\mathbb{Z}}$; we simply replace all index sets of the type $1, 2, \ldots n$ by $-n, \ldots n$.

Let us shortly recall the standard setup for shift-invariant spaces [1]. Let $\psi \in L^2(\mathbb{R})$ be a continuous function for which the following conditions are satisfied:

- for some $C > 0, s > 1$,

$$|\psi(x)| \leqslant C(1+|x|)^{-s}. \tag{22}$$

- $\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a Riesz basis for its closed linear span,

$$\mathcal{H} := \overline{\text{span}}\{\psi(\cdot - k)\}_{k\in\mathbb{Z}} = \left\{ \sum_{k\in\mathbb{Z}} c_k \psi(\cdot - k) \mid \{c_k\} \in \ell^2 \right\}. \tag{23}$$

Then $\mathcal{H}$ is a so-called reproducing kernel Hilbert space, i.e., the point evaluations $f \mapsto f(x)$ are continuous linear functionals on $\mathcal{H}$, [15,4]. Thus, for each $x \in \mathbb{R}$, there exists $K_x \in \mathcal{H}$ such that

$$f(x) = \langle f, K_x \rangle. \tag{24}$$

If $\{\lambda_k\}_{k\in\mathbb{Z}}$ is a set of sampling for $\mathcal{H}$, i.e., there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leqslant \sum_{k\in\mathbb{Z}} |f(\lambda_k)|^2 \leqslant B\|f\|^2 \;\forall f \in \mathcal{H},$$

then $\{K_{\lambda_k}\}_{k\in\mathbb{Z}}$ is a frame for $\mathcal{H}$. Our goal is to compute the dual frame $\{\tilde{K}_{\lambda_k}\}_{k\in\mathbb{Z}}$.

Let φ be the function whose Fourier transform is given by

$$\hat{\varphi}(\omega) = \frac{\hat{\psi}(\omega)}{\sqrt{\sum_{k\in\mathbb{Z}} |\hat{\psi}(\omega+k)|^2}}; \tag{25}$$

then $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $\mathcal{H}$, see [4,6]. Furthermore $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ inherits the localization properties of $\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$, which is an immediate consequence of Lemma 3.1 in [8] (see also [13]) and the square-root theorem for Banach algebras [7]. Formula (25) provides an efficient and stable way to approximate φ numerically, simply by truncating the sum in (25) and carrying out the inverse Fourier transform by standard numerical integration (e.g., by using an FFT applied to $\{\hat{\varphi}(k/N)\}_{k=-N/2}^{N/2}$ for N large enough).

Using the orthonormal basis $\{\overline{\varphi(\cdot - k)}\}_{k\in\mathbb{Z}}$ for $\mathcal{H}$, it is known [1] that K_{λ_k} can be written as

$$K_{\lambda_k} = \sum_{n\in\mathbb{Z}} \overline{\varphi(\lambda_k - n)}\varphi(\cdot - n). \tag{26}$$

It follows from paragraph 3.3 in [8] that $\{K_{\lambda_k}\}_{k\in\mathbb{Z}}$ is s-localized with respect to $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$.

Let S denote the frame operator for $\{K_{\lambda_k}\}_{k\in\mathbb{Z}}$. The dual frame is given by $\tilde{K}_{\lambda_j} = S^{-1}K_{\lambda_j}$, $j \in \mathbb{Z}$; in order to apply Theorem 2.8 with $f = K_{\lambda_j}$, we have to calculate $z := \Lambda_n^{-1}T_n^* K_{\lambda_j}$, i.e., to solve the equation

$$\Lambda_n z = T_n^* K_{\lambda_j}. \tag{27}$$

The entries of the matrix Λ_n w.r.t. the choice of orthonormal basis $e_k = \varphi(\cdot - k)$ are

$$\begin{aligned}
\langle S_{m(n)} e_k, e_l\rangle &= \sum_{j=-m(n)}^{m(n)} \langle e_k, K_{\lambda_j}\rangle\langle K_{\lambda_j}, e_l\rangle \qquad (28)\\
&= \sum_{j=-m(n)}^{m(n)} \left\langle \varphi(\cdot - k), \sum_{n\in\mathbb{Z}} \overline{\varphi(\lambda_j - n)}\varphi(\cdot - n)\right\rangle \\
&\quad \times \left\langle \sum_{m\in\mathbb{Z}} \overline{\varphi(\lambda_j - m)}\varphi(\cdot - m), \varphi(\cdot - l)\right\rangle \\
&= \sum_{j=-m(n)}^{m(n)} \varphi(\lambda_j - k)\overline{\varphi(\lambda_j - l)}, \qquad (29)
\end{aligned}$$

where we have used that $\langle \varphi(\cdot - k), \varphi(\cdot - l)\rangle = \delta_{k,l}$. Via (24),

$$\begin{aligned}
T_n^* K_{\lambda_j} &= \{\langle K_{\lambda_j}, \varphi(\cdot - l)\rangle\}_{l=-m(n)}^{l=m(n)} \\
&= \left\{\overline{\varphi(\lambda_j - l)}\right\}_{l=-m(n)}^{l=m(n)}.
\end{aligned}$$

Now Eq. (27) can be solved by standard methods from linear algebra, such as conjugate gradient type techniques. Theorem 2.8 implies that in this way we can approximate the dual frame with an error that decreases polynomially for $n \to \infty$.

Our approach also provides an answer to another computational issue, which appears in, e.g., [1]:

Example 3.1. Let $\{\lambda_j\}_{j\in\mathbb{Z}}$ be a set of sampling and assume that we want to reconstruct $f \in \mathcal{H}$ from the samples $\{f(\lambda_j)\}_{j\in\mathbb{Z}}$. Here we can assume without loss of generality that the functions $\{\varphi(\cdot - k)\}$ that decay as in (22) and span $\mathcal{H}$ as in (23) form an ONB instead of a Riesz basis (otherwise we can always transform the Riesz basis into an ONB with the same decay properties as described in (25)). Since f can be written as $f = \sum_{k\in\mathbb{Z}} c_k\varphi(\cdot - k)$ we can reconstruct f by computing the coefficient vector $c = \{c_k\}_{k\in\mathbb{Z}}$, which in turn can be calculated by solving the infinite-dimensional system of equations

$$Uc = \{f(\lambda_j)\}_{j\in\mathbb{Z}}, \tag{30}$$

where U is a biinfinite matrix with entries

$$U_{j,k} = \varphi(\lambda_j - k),\ j,k \in \mathbb{Z}. \tag{31}$$

Of course in reality we cannot solve an infinite-dimensional system but we have to come up with a finite-dimensional system instead. No statement is made in [1] about how to solve (30) in practice; in fact the finite section method does in general not apply to (30) (see Example 3.2 below). However, we now show that the finite section method applies to the normal equations

$$U^*Uc = U^*\{f(\lambda_j)\}_{j\in\mathbb{Z}}. \tag{32}$$

Note that

$$(U^*U)_{k,l} = \sum_{j\in\mathbb{Z}} \overline{\varphi(\lambda_j - k)}\varphi(\lambda_j - l), \quad k,l \in \mathbb{Z}; \tag{33}$$

by a computation as in (28)–(29) this shows that U^*U coincides with the complex conjugated of the matrix $\{\langle Se_k, e_l\rangle\}_{k,l\in\mathbb{Z}}$. Thus if we approximate U^*U by the matrix $\{\lambda_{k,l}\}_{k,l=-n}^n$ with the entries in (29), we can indeed stably approximate the coefficients $\{c_k\}_{k\in\mathbb{Z}}$ and thus numerically reconstruct f with an approximation error governed by the decay rate of φ.

Here is a concrete example where the finite section method does not apply to (30):

Example 3.2. Let $\psi = (2 - 4\,|x|)\chi_{[-1/2,1/2]}$. Then $\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a Riesz basis for its closed span which is denoted by $\mathcal{H}$. Furthermore, let $\lambda_{2j} = 2j - 1$, $\lambda_{2j-1} = 2j$ for $j \in \mathbb{Z}$. Then any $f \in \mathcal{H}$ can be written as $f = \sum_{k\in\mathbb{Z}} c_k\psi(\cdot - k)$ for $c = \{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, in particular $f(\lambda_j) = \sum_{k\in\mathbb{Z}} c_k\psi(\lambda_j - k) = c_{\lambda_j}$. Thus

$$\sum_{j\in\mathbb{Z}} |f(\lambda_j)|^2 = \sum_{k\in\mathbb{Z}} |c_{\lambda_j}|^2 = \sum_{k\in\mathbb{Z}} |c_k|^2, \tag{34}$$

because $\{\lambda_j\}_{j\in\mathbb{Z}}$ is just a reordering of $\mathbb{Z}$. By the Riesz basis condition this implies that $\{\lambda_j\}_{j\in\mathbb{Z}}$ is a set of sampling for $\mathcal{H}$, hence theoretically any function $f \in \mathcal{H}$ can be reconstructed from its samples $\{f(\lambda_j)\}_{j\in\mathbb{Z}}$. Considering (30) with the matrix U in (31) we note that the finite section method obviously does not work for the natural choice of orthonormal basis: in fact, the finite sections U_n of the matrix U are of the form

$$U_0 = \psi(\lambda_0 - 0) = 0, \quad U_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, U_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$U_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence none of the U_n is invertible, although U is invertible, since it is just a permutation matrix.

In contrast, the finite section method described in Example 3.1 works very well in this case (of course the following steps are not really necessary in this case, since U represents a unitary operator): The family $\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$ can be transformed into an ONB $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ where φ has exponential decay (the statement about the decay is some kind of folklore result, which is not stated explicitly in the literature; it follows by extending the results in [13] to $L^2(\mathbb{R})$ or by combining Lemma 4.1 in [8] with the square root theorem for Banach algebras [7]). Hence, by the derivations in Example 3.1 and by Theorem 2.8 the finite section method applies to the normal equations (32) with an exponential order of convergence.

4. The Casazza–Christensen method

As a final application of our results we now prove that they lead to an improvement of the Casazza–Christensen method (cf. [3]) for approximation of the inverse frame operator related to a general frame. We consider again a frame $\{f_k\}_{k=1}^\infty$ for a Hilbert space $\mathcal{H}$, the associated frame operator defined in (1), a sequence of subspaces of $\mathcal{H}$ as in (3), and the associated orthogonal projections P_n.

A straightforward application of Theorem 1.10 in [10] shows that

$$(P_n S P_n)^{-1} P_n f \to S^{-1} f \ \forall f \in \mathcal{H}.$$

However, in order to obtain a practically applicable result we have to replace the operators $P_n S P_n$ by operators which only involve a finite number of the frame elements. Given $n \in \mathbb{N}$, consider again the frame operator S_n associated to $\{f_k\}_{k=1}^n$, see (8). S_n is invertible on $\mathcal{H}_n$, but usually $S_n^{-1} P_n f$ does not converge to $S^{-1} f$. Our purpose is to show that for $n \in \mathbb{N}$ we can chose $m(n) \in \mathbb{N}$ such that

$$(P_n S_{n+m(n)} P_n)^{-1} P_n f \to S^{-1} f \text{ as } n \to \infty \ \forall f \in \mathcal{H}. \tag{35}$$

The possibility of doing so is also proved in [3], but the method presented here leads to considerably smaller values for $m(n)$, a very important issue as soon as the computational effort is considered.

Theorem 4.1. *Choose* $R < A$. *Given* $n \in \mathbb{N}$, *choose* $m(n) \in \mathbb{N}$ *such that*

$$\sum_{k=n+m(n)+1}^{\infty} |\langle f, f_k\rangle|^2 \leqslant R\|f\|^2 \ \forall f \in \mathcal{H}_n.$$

Then $\{P_n S_{n+m(n)} P_n\}_{n=1}^{\infty}$ *is applicable*; *in particular,* (35) *holds.*

Proof. Regardless of the choice of $m(n) \geqslant 0$ the sequence $\{P_n S_{n+m(n)} P_n\}_{n=1}^{\infty}$ is an approximation method for S. Now

$$P_n S_{n+m(n)} P_n = P_n S P_n + P_n(S_{n+m(n)} - S)P_n;$$

thus we can consider $\{P_n S_{n+m(n)} P_n\}_{n=1}^{\infty}$ as a perturbation of the stable approximation method $\{P_n S P_n\}_{n=1}^{\infty}$. For all $n \in \mathbb{N}$,

$$\begin{aligned}\left\| P_n(S_{n+m(n)} - S)P_n \right\| &= \sup_{\|f\|=1,\, f\in\mathcal{H}_n} |\langle P_n(S_{n+m(n)} - S)P_n f, f\rangle| \\ &= \sup_{\|f\|=1,\, f\in\mathcal{H}_n} \sum_{k=n+m(n)+1}^{\infty} |\langle f, f_k\rangle|^2 \\ &\leqslant R \\ &< A.\end{aligned}$$

By Lemma 2.1 we have $\|(P_n S P_n)^{-1}\| \leqslant 1/A$ for all n, so it follows that

$$\sup_n \left\| P_n(S_{n+m(n)} - S)P_n \right\| \leqslant \inf_n \left\| (P_n S P_n)^{-1} \right\|^{-1}.$$

By Lemma 1.3 we conclude that $(P_n S_{n+m(n)} P_n)$ is applicable. □

Compared to the result by Casazza/Christensen [3], the advantage of Theorem 4.1 is that R can be chosen as any constant smaller than A: in [3] a similar result was obtained, but with R was depending on n, and forced to tend to zero for $n \to \infty$. This, in turn, implies that $m(n)$ is forced to be unnecessarily large, and thereby complicate the computations.

Final remark. All results for approximating S^{-1} can be extended to approximating $S^{-\frac{1}{2}}$, for instance by proceeding along similar lines as in Theorem 8.1.4 of [5]. This extension is useful when one wants to numerically compute tight frames of the form $\{S^{-\frac{1}{2}} f_k\}_{k=1}^{\infty}$. Furthermore, using the results in [2] one can easily extend the results in this paper to frames whose localization is characterized by decay other than polynomial or exponential decay.

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